THE PLURIPOLAR HULL OF A GRAPH AND FINE ANALYTIC CONTINUATION

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Abstract. We show that if the graph of a bounded analytic function in the unit disk $\mathbb{D}$ is not complete pluripolar in $\mathbb{C}^2$ then the projection of the closure of its pluripolar hull contains a fine neighborhood of a point $p \in \partial\mathbb{D}$. On the other hand we show that if an analytic function f in $\mathbb{D}$ extends to a function $\mathcal{F}$ which is defined on a fine neighborhood of a point $p \in \partial\mathbb{D}$ and is finely analytic at p then the pluripolar hull of the graph of f contains the graph of $\mathcal{F}$ over a smaller fine neighborhood of p. We give several examples of functions with this property of fine analytic continuation. As a corollary we obtain new classes of analytic functions in the disk which have non-trivial pluripolar hulls, among them C^∞ functions on the closed unit disk which are nowhere analytically extendible and have infinitely-sheeted pluripolar hulls. Previous examples of functions with non-trivial pluripolar hull of the graph have fine analytic continuation.

1. Introduction

A subset E of a domain $\Omega \subset \mathbb{C}^N$ is called pluripolar in Ω, if for all $z \in E$ there exist a connected neighborhood U_z of z in Ω and a plurisubharmonic function $u(z,w) \not\equiv -\infty$ defined on U_z such that

$$E \cap U_z \subset \{(z,w) \in U_z : u(z,w) = -\infty\}.$$

By Josefson's theorem (see [Jos]), a set $E \subset \mathbb{C}^N$ is pluripolar if and only if there exists a globally defined plurisubharmonic function $u(z,w)$ such that

$$E \subset \{(z,w) \in \mathbb{C}^N : u(z,w) = -\infty\}.$$

In other words a pluripolar set is a subset of the $-\infty$-locus of a globally defined plurisubharmonic function. Pluripolar sets are the exceptional sets in pluripotential theory. This motivates the interest in understanding the structure of pluripolar sets. A set $E \subset \Omega$ is called complete pluripolar in Ω if E is the exact $-\infty$-locus of a plurisubharmonic function defined in Ω. On the contrary, some subsets $E \subset \Omega$ (e.g. proper open subsets of connected analytic submanifolds) have the property that any plurisubharmonic function which is $-\infty$ on E is $-\infty$ on a larger set. This leads to the notion of the *pluripolar hull* E^*_Ω (see [LePo]) of a pluripolar subset $E \subset \Omega$,

$$E^*_\Omega \stackrel{def}{=} \bigcap \{z \in \Omega : u(z) = -\infty\},$$

where the intersection is taken over *all* plurisubharmonic functions in Ω which equal $-\infty$ on E. In general, it is difficult to describe the pluripolar hull of a given set E. Initiated by a paper of Saddulaev ([Sad]) the pluripolar hull of graphs of certain analytic functions has been studied in a number of papers (see e.g. [EdWi1],[EdWi2],[EdWi3], [LePo], [Sic1],[Sic2],[Wie1] and [Wie2]).

For a subset A of the complex plane $\mathbb{C}$ and a complex valued function f on A we denote by $\Gamma_f(A)$ the graph of f over A,

$$\Gamma_f(A) = \{(z,w) \in \mathbb{C}^2 : z \in A, w = f(z)\}.$$

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Let f be a holomorphic function in the unit disk $\mathbb{D} \subset \mathbb{C}$. Clearly $\Gamma_f(\mathbb{D})$ is a pluripolar set. It is a natural attempt to relate non-triviality of the pluripolar hull of $\Gamma_f(\mathbb{D})$ to the existence of analytic continuation or various kinds of generalized analytic continuation of f across some part of $\partial\mathbb{D}$. In [LeMaPo] Levenberg, Martin and Poletsky conjectured that if f is analytic in $\mathbb{D}$ and f does not extend holomorphically across $\partial\mathbb{D}$, then $\Gamma_f(\mathbb{D})$ is complete pluripolar. This conjecture was disproved in [EdWi2]. In [Sic1], Siciak noticed that the function in [EdWi2] possesses pseudocontinuation across a subset of the circle of full measure and showed that the pluripolar hull of its graph contains the graph of the pseudocontinuation. He also noticed that if an analytic function f in $\mathbb{D}$ admits pseudocontinuation through a set E of positive measure on the circle and the graph of the non-tangential limits is in the pluripolar hull of $\Gamma_f(\mathbb{D})$ then also the graph of the pseudocontinuation is in the mentioned pluripolar hull. In [Sic1] he showed by an example that the existence of pseudocontinuation of the function f is not necessary for non-triviality of the pluripolar hull of $\Gamma_f(\mathbb{D})$.

The notion of fine analytic continuation seems to us better adapted to understand pluripolar completeness of graphs.

Recall that the fine topology was introduced by Cartan (see e.g. [Bre]) as the weakest topology for which all subharmonic functions are continuous. A neighborhood basis of a point in this topology consists of sets which differ from a Euclidean neighborhood of this point by a set which is thin at this point. Thin sets were introduced by Brelot. A set $F \subset \mathbb{C}$ is thin at a point ξ, if either ξ is not in its closure $\overline{F}$ or $\xi \in \overline{F}$ and there exists a subharmonic function $\mathcal{V}$ in a neighborhood of ξ such that $\overline{\lim}_{z\in F, z\to\xi}\mathcal{V}(z) < \mathcal{V}(\xi)$. One can always choose $\mathcal{V}$ in such a way that the limit on the left equals $-\infty$. For a point $p \in \mathbb{C}$ and a positive number r we denote by $D(p,r)$ the open disk of radius r and center p.

By a closed fine neighborhood V of p we mean a connected closed set which has the form $B \setminus U$ for some connected closed set B and an open set $U \subset \mathbb{C}$ which is thin at p. Note that U can be taken simply connected. We will often consider $B = \overline{D(p,r)}$ for some $r > 0$.

Definition 1. A continuous function $\mathcal{F}$ on a closed fine neighborhood V of a point $p \in \mathbb{C}$ is called finely analytic at p (on V) if $\mathcal{F}$ can be approximated uniformly on V by analytic functions $\mathcal{F}_n$ in a neighborhood $U(\mathcal{F}_n)$ of V.

We will say that a continuous function on a subset S of $\mathbb{C}$ has the Mergelyan property if it can be approximated uniformly on S by analytic functions in a neighborhood of S. Mergelyan's Theorem states that for compact sets K with finitely many components of the complement all continuous functions on K which are holomorphic in the interior $\mathrm{Int}K$ have this property.

Note that the term finely analytic functions is well known and is used for functions which have the Mergelyan property on a finely open set. Here we consider only the local definition above (we do not require that V contains a fine neighborhood of each of its points). Even in this local situation a weak version of the unique continuation property holds (see Proposition 3 below).

Definition 2. Suppose f is analytic in the unit disk $\mathbb{D}$. Let p be a point on the unit circle $\partial\mathbb{D}$. We say that f has fine analytic continuation $\mathcal{F}$ at p if there exists a closed fine neighborhood V of p such that $V \cap \mathbb{D} \supset \overline{D(p,r)} \cap \mathbb{D}$ for some $r > 0$, and a finely analytic function $\mathcal{F}$ at p on V such that $\mathcal{F}|_{\mathbb{D}\cap V} = f$.

Remark 1. The conditions of Definition 2 are satisfied, in particular, if $V = \overline{D(p,r)} \setminus U$, where $r > 0$, $U \subset \mathbb{C} \setminus \overline{\mathbb{D}}$ is open and thin at p and $\mathcal{F} = \mathcal{G} + \mathcal{C}$ on V, where $\mathcal{G}$ is analytic on $D(p,r)$ and continuous on $\overline{D(p,r)}$, and $\mathcal{C}$ is the Cauchy-type integral of a finite Borel measure μ concentrated on U such that for an increasing sequence of compacts $\kappa_n \subset U$

the functions

$$\mathcal{C}_n(z) = -\frac{1}{\pi}\int_{\kappa_n} \frac{d\mu(\xi)}{\xi - z}, \qquad z \notin \kappa_n,$$

converge uniformly to $\mathcal{C}(z)$ on V.

Note that in Definition 2 we do not require that the set $V \setminus \overline{\mathbb{D}}$ has interior points. Nevertheless, by the mentioned weak unique continuation property, if fine analytic continuation to a given set V exists then it is unique on a, maybe, smaller fine neighborhood.

Here are examples of analytic functions in the unit disk which allow fine analytic continuation at certain point of the unit circle.

Example 1. The functions constructed in [Sic1] and [EdWi2] satisfy the definition. Indeed, they have the following form. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of points contained in $\mathbb{C} \setminus \overline{\mathbb{D}}$ which cluster to a subset of $\partial\mathbb{D}$ and do not have other cluster points. Let $D(a_n, \rho_n) \subset \mathbb{C} \setminus \overline{\mathbb{D}}$ be a sequence of pairwise disjoint disks around a_n of radius $\rho_n > 0$ such that $U \overset{def}{=} \bigcup_{n=1}^{\infty} D(a_n, \rho_n)$ is thin at a point $p \in \partial\mathbb{D}$. Let c_n be a sequence of complex numbers such that $\sum |c_n| < \infty$ and $|c_n| \leq (1/n^2)\rho_n$. Define

$$f(z) = \sum_{n=1}^{\infty} \frac{c_n}{z - a_n}, \qquad z \in \mathbb{C} \setminus \bigcup_{n=1}^{\infty} D(a_n, \rho_n).$$

It is immediate to check that the conditions of Definition 2 are satisfied.

Example 2. Let $U \subset \mathbb{C} \setminus \overline{\mathbb{D}}$ be open, relatively compact and thin at every point of the unit circle $\partial\mathbb{D}$, and suppose moreover that $\mathbb{C} \setminus U$ is connected. Such sets can easily be obtained by first choosing points p_n which accumulates to each point of $\partial\mathbb{D}$ and to no other point, then choosing $\rho_n > 0$ such that the disks $D(p_n, \rho_n)$ are pairwise disjoint and do not meet $\overline{\mathbb{D}}$. Choosing then $a_n > 0$ so that the series $\sum a_n \log(|z - p_n|/\rho_n)$ converges to a subharmonic function which is non-negative outside $\bigcup D(p_n, \rho_n)$ and, finally, choosing $r_n > 0$ such that the function is less than -1 on $U \overset{def}{=} \bigcup D(p_n, r_n)$. Let $\mathcal{F}$ be a C^1 function on $\widehat{\mathbb{C}}$ (here $\widehat{\mathbb{C}}$ denotes the Riemann sphere), such that $\overline{\partial}\mathcal{F} = 0$ on $\mathbb{C} \setminus U$. By the Cauchy-Green's formula

$$\mathcal{F}(z) = -\frac{1}{\pi}\iint_U \frac{\overline{\partial}\mathcal{F}}{\xi - z} dm_2(\xi) + \mathcal{F}(\infty).$$

Since the density $\overline{\partial}\mathcal{F}$ is bounded and the Cauchy kernel is locally integrable with respect to two-dimensional Lebesgue measure, the function $f = \mathcal{F}|_{\mathbb{D}}$ has fine analytic extension at each point $p \in \partial\mathbb{D}$, moreover, $\mathcal{F}$ has the Mergelyan property on $\widehat{\mathbb{C}} \setminus U$.

More generally, let U be as described and let $g \in L^{2+\epsilon}(\mathbb{C})$ for some $\epsilon > 0$ and $g = 0$ outside U. The function

$$\mathcal{F}(z) \overset{def}{=} -\frac{1}{\pi}\iint_{\mathbb{C}} \frac{g(\xi) dm_2(\xi)}{\xi - z},$$

satisfies the condition of Definition 2 and has the Mergelyan property on $\widehat{\mathbb{C}} \setminus U$. If g is a C^∞ function then $\mathcal{F}$ is a C^∞ function on the whole Riemann sphere. If g is continuous and in addition $g \geq 0$ and $g > 0$ at some point of each connected component of U, then $f = \mathcal{F}|_{\mathbb{D}}$ does not have analytic extension across any arc of $\partial\mathbb{D}$. Indeed, suppose it has analytic continuation f_p to a disk $D(p, r)$ for some $p \in \partial\mathbb{D}$. By Proposition 3 below f_p coincides with the fine analytic continuation $\mathcal{F}$ on some fine neighborhood V_1 of p. V_1 contains a circle $\partial D(p, \rho)$, $0 < \rho < r$. (This is well known. The reader who is not familiar with potential theory will find a proof below in Section 2.) Let $\kappa_n \subset U$ be an exhausting sequence of compact subsets of U. Since $V_1 \subset \mathbb{C} \setminus U$ for each n,

$\kappa_n \cap \partial D(p,\rho) = \emptyset$ and

$$\begin{aligned} 0 &= \int_{|z-p|=\rho} f_p dz = \lim_{n\to\infty} -\frac{1}{\pi}\int_{|z-p|=\rho} dz \iint_{\kappa_n} \frac{\overline{\partial}\mathcal{F}}{\xi - z} dm_2(\xi) = \\ &= \lim_{n\to\infty} -\frac{1}{\pi}\iint_{\kappa_n} dm_2(\xi)\overline{\partial}\mathcal{F}(\xi)\int_{|z-p|=\rho}\frac{1}{\xi-z}dz = \\ &= \lim_{n\to\infty} \frac{2\pi i}{\pi}\iint_{\kappa_n} dm_2(\xi)\overline{\partial}\mathcal{F}(\xi)\cdot\chi_{D(p,\rho)}(\xi) \neq 0. \end{aligned}$$

Here $\chi_{D(p,\rho)}$ is the characteristic function of the disk $D(p,\rho)$. The contradiction proves the assertion.

Example 3. The third example is related to pseudocontinuation across certain subsets of positive length of the unit circle.

Definition 3. A function f which is analytic in $\mathbb{D}$ is said to have *pseudocontinuation* from $\mathbb{D}$ across the set $\mathcal{E} \subset \partial\mathbb{D}$ of positive measure to a domain $D_e \subset \{z \in \mathbb{C} : |z| > 1\}$ if for all $z \in \mathcal{E}$ the domain D_e contains truncated non-tangential cones with vertices at z, and there exists an analytic function $\widetilde{f}$ in D_e, such that f and $\widetilde{f}$ have the same non-tangential limits at z. In this case we call $\widetilde{f}$ the pseudocontinuation of f.

For convenience we will specify the situation in the following way. We will restrict ourselves to the case where $\mathcal{E}$ is closed and the angles and the diameters of the truncated non-tangential cones in D_e are uniformly bounded from below by positive constants. Shrinking perhaps $\mathcal{E}$ and D_e we may assume that D_e is a bounded domain, moreover, that it consists of the union of all open truncated non-tangential cones with symmetry axes orthogonal to the circle and that all the mentioned cones have the same angle and the pseudocontinuation is continuous in $\overline{D_e}$. So, D_e has the shape of a "saw" near $\mathcal{E}$. Replace $\mathbb{D}$ by a domain $D_i \subset \mathbb{D}$ which is symmetric to D_e in a small neighborhood of $\mathcal{E}$ in such a way that the bounded components of the complement of $\overline{D_i} \cup \overline{D_e}$ are similar rhombs $\Diamond_l$ containing the complementary arcs of $\mathcal{E}$ in the circle. (The endpoints of one of the symmetry axes of the rhomb $\Diamond_l$ are the endpoints of a connected component of $\partial\mathbb{D} \setminus \mathcal{E}$.) Assume that the unbounded component of $\mathbb{C} \setminus (\overline{D_e} \cup \overline{D_i})$ intersects $\partial\mathbb{D}$ along a connected arc. We may assume that the construction is made so that the original function is continuous in $\overline{D_i}$.

The original function together with its pseudocontinuation across $\mathcal{E}$ defines a continuous function on $\overline{D_i} \cup \overline{D_e}$, which is analytic in $D_i \cup D_e$. Denote the space of such functions by $A(\overline{D_i} \cup \overline{D_e})$.

It will be convenient to give the third example with $\mathbb{D}$ replaced by D_i. It can be stated for $\mathbb{D}$ instead with obvious changes.

Proposition 1. *Let D_i, D_e and $\mathcal{E}$ be as above and let $f \in A(\overline{D_i} \cup \overline{D_e})$. Put $U = \mathbb{C} \setminus (\overline{D_i} \cup \overline{D_e})$. If U is thin at a point $p \in \mathcal{E}$ and f is Hölder continuous of order $\alpha \in (0,1]$, then $f|_{D_i}$ has fine analytic continuation $f|_{V_1}$ at p for a fine neighborhood $V_1 \subset \overline{D_i} \cup \overline{D_e}$ of p.*

Note that the fact that $\mathcal{E}$ has positive length follows from the fact that its complement in $\partial\mathbb{D}$ is thin at p.

We will prove Proposition 1 in Section 2. The following Theorem holds.

Theorem 1. *Let f be analytic in $\mathbb{D}$ and let $p \in \partial\mathbb{D}$. Suppose f has fine analytic continuation $\mathcal{F}$ at p to a closed fine neighborhood V of p. Then there exists another closed fine neighborhood $V_1 \subset V$ of p, such that the graph $\Gamma_{\mathcal{F}}(V_1)$ is contained in the pluripolar hull of $\Gamma_f(\mathbb{D})$.*

Moreover, if $IntV \setminus \overline{\mathbb{D}}$ has a connected component $\overset{\circ}{V}$ which is not thin at p then $\Gamma_{\mathcal{F}}(\overset{\circ}{V})$ is contained in the pluripolar hull of $\Gamma_f(\mathbb{D})$.

If $V = \overline{D(p,r)} \setminus U$ with U open and thin at p and $\overline{U} \setminus \overline{\mathbb{D}}$ is also thin at p then there exists a unique connected component of $IntV \setminus \overline{\mathbb{D}} = \{|z-p| < r, |z| > 1\} \setminus \overline{U}$ which is not thin at p.

Note that Theorem 1 holds as well in the situation of Proposition 1 with $\mathbb{D}$ replaced by D_i. Theorem 1 can be slightly generalized.

Theorem 2. *Let $\mathcal{F}$ be finely analytic on a closed fine neighborhood $V = \overline{D(p,r)} \setminus U$ of a point $p \in \mathbb{C}$. Let γ be a smooth arc, $\gamma : [-1,1] \to \mathbb{C}$ with $\gamma(0) = p$, which divides $D(p,r)$ into two components $D_+(p,r)$ and $D_-(p,r)$. Suppose $\overline{U} \setminus \gamma$ is thin at p. Then there are unique connected components V_+ and V_- of $D_+(p,r) \setminus \overline{U}$ and $D_-(p,r) \setminus \overline{U}$ which are not thin at p. For those we have that the pluripolar hull of $\Gamma_{\mathcal{F}}(V_+)$ is contained in $\Gamma_{\mathcal{F}}(V_-)$ and vice versa.*

Corollary 1. *Let D_i, D_e and $\mathcal{E}$ be as described before Proposition 1 and let $f \in A(\overline{D_i} \cup \overline{D_e})$. If $U = \mathbb{C} \setminus (\overline{D_i} \cup \overline{D_e})$ is thin at some point $p \in \mathcal{E}$ and f is Hölder continuous of order $\alpha \in (0,1]$ then $\Gamma_f(D_e)$ is contained in the pluripolar hull of $\Gamma_f(D_i)$.*

Theorem 2 and Corollary 1 has the following consequence.

Corollary 2. *There exist univalent analytic functions in the disk which are smooth up to the boundary and are nowhere analytically continuable and have an analytic manifold in the non-trivial part of the pluripolar hull of the graph.*

Functions with the mentioned property except univalency were constructed in [EdWi2]. Theorem 1 and Example 2 give further classes of functions with the mentioned property (not necessarily univalent functions). The present constructions are simpler then those in [EdWi2].

Denote by π_1 the projection onto the first coordinate plane in $\mathbb{C}^2$, $\pi_1(z_1, z_2) = z_1$ for $z = (z_1, z_2) \in \mathbb{C}^2$. Theorem 1 and its corollaries have the following counterpart.

Theorem 3. *Let f be analytic in $\mathbb{D}$. Suppose $(\Gamma_f(\mathbb{D}))^*_{\mathbb{C}^2}$ is not contained in $\mathbb{D} \times \mathbb{C}$. Put $E \stackrel{def}{=} \overline{(\Gamma_f(\mathbb{D}))^*_{\mathbb{C}^2}}$. Then $\pi_1(E)$ contains a fine neighborhood of a point $p \in \partial\mathbb{D}$ (i.e. $\mathbb{C} \setminus \pi_1(E)$ is thin at p).*

For bounded analytic functions the following sharper version of Theorem 3 holds.

Theorem 4. *Let f be a bounded analytic function in $\mathbb{D}$ whose graph is not complete pluripolar in $\mathbb{C}^2$. Then $\pi_1(\overline{(\Gamma_f(\mathbb{D}))^*_{\mathbb{C}^2}})$ contains a fine neighborhood of a point $p \in \partial\mathbb{D}$.*

The reduction of Theorem 4 to Theorem 3 follows immediately from results in [EdWi3]. Lemma 2 in Section 3 shows that Theorem 3 can be slightly improved. We will not give the corresponding statements here.

Corollary 3. *Let $\mathcal{D} = \mathbb{D} \cup D$, where D is a domain, $D \subset \mathbb{C} \setminus \overline{\mathbb{D}}$, which contains a truncated non-tangential cone of fixed size with vertex z for each $z \in \mathcal{E}$, $\mathcal{E}$ a closed subset of positive length on $\partial\mathbb{D}$. Let f be holomorphic in $\mathcal{D}$ and continuous in $\overline{\mathcal{D}} = \overline{\mathbb{D}} \cup \overline{D}$ (hence $f|_{\mathbb{D}}$ and $f|_D$ are pseudocontinuations of each other across the set $\mathcal{E}$). Suppose $\mathbb{C} \setminus \overline{\mathcal{D}}$ is non-thin at every point $z \in \partial\mathbb{D}$ and $\Gamma_f(\overline{\mathcal{D}})$ is complete pluripolar. Then $\Gamma_f(\mathbb{D})$ is complete pluripolar.*

Note that functions f with the mentioned properties exist (see [Edl], to appear). Corollary 3 states roughly that if two analytic manifolds in $\mathbb{C}^2$ have contact along a set which is not massive enough in potential theoretic terms then the property of plurisubharmonic functions in $\mathbb{C}^2$ to be $-\infty$ does not propagate from one of the manifolds to the other one.

Proof of Corollary 3. $E = \overline{(\Gamma_f(\mathbb{D})^*_{\mathbb{C}^2})} \subset \Gamma_f(\overline{\mathcal{D}})$ since the latter set is closed and complete pluripolar. Hence $\pi_1(E) \subset \overline{\mathcal{D}}$ and $\mathbb{C} \setminus \pi_1(E) \supset \mathbb{C} \setminus \overline{\mathcal{D}}$ is non-thin at any point $p \in \partial\mathbb{D}$. $\square$

Note that the set V in Theorem 1 and the set E in Theorem 3 were assumed to be closed. It would be interesting to remove the condition of closeness. We conclude this Section with an example of fine analytic continuation to a set with no interior points outside the unit disk, with an example with infinitely sheeted pluripolar hull and with some open problems.

Problem 1. Is $\pi_1((\Gamma_f(\mathbb{D}))^*_{\mathbb{C}^2})$ finely open ?

Problem 2. Let f be analytic in $\mathbb{D}$, not necessarily bounded. Suppose that the pluripolar hull of $\Gamma_f(\mathbb{D})$ is contained in $\mathbb{D}\times\mathbb{C}$. Does this imply that $\Gamma_f(\mathbb{D})$ is complete pluripolar in $\mathbb{C}^2$?

Example 4. Consider all points in $\mathbb{C}\setminus\overline{\mathbb{D}}$ with rational coordinates. This set is countable and accumulates, in particular to the whole circle $\partial\mathbb{D}$. As in Example 2 there exists a subharmonic function $\mathcal{U}$ which is $-\infty$ on this set and non-negative at each point $p\in\overline{\mathbb{D}}$. Let U be the set of points on which $\mathcal{U}<-1$. U is open, contained in $\mathbb{C}\setminus\overline{\mathbb{D}}$ and it is thin at each point of the unit circle. Moreover $\mathbb{C}\setminus U$ is connected, i.e. U is simply connected which is a consequence of the maximum principle.

Let $\mathcal{F}$ be a continuous function on $\mathbb{C}\setminus U$ which has the Mergelyan property on each compact subset of $\mathbb{C}\setminus U$ and has complete pluripolar graph $\Gamma_{\mathcal{F}}(\mathbb{C}\setminus U)$. Such functions were constructed in [Edl] for arbitrary closed subsets of $\mathbb{C}$. Then $f=\mathcal{F}|_{\mathbb{D}}$ is analytic and f has fine analytic continuation at each point $p\in\partial\mathbb{D}$. Hence by Theorem 2 there exists a set A_1 which contains a fine neighborhood of each point of the circle, such that the graph $\Gamma_{\mathcal{F}}(A_1)$ is in the pluripolar hull of $\Gamma_{\mathcal{F}}(\mathbb{D})$. However the mentioned pluripolar hull is contained in $\Gamma_{\mathcal{F}}(\mathbb{C}\setminus U)$, and the set $\mathbb{C}\setminus U$ does not have interior points outside the unit disk $\mathbb{D}$. Hence in this case the non-trivial part of the pluripolar hull of $\Gamma_f(\mathbb{D})$ does not have analytic structure, i.e. there is no piece of a non-trivial analytic manifold contained in it. The following problems arise.

Problem 3. Let f be analytic in $\mathbb{D}$. Suppose $\pi_1((\Gamma_f(\mathbb{D}))^*_{\mathbb{C}^2}$ has an interior point p outside $\mathbb{D}$. (Note that here we consider the pluripolar hull itself, not its closure!) Is there a neighborhood V_p of p in $\mathbb{C}$ and a relatively closed subset X of $V_p\times\mathbb{C}$ such that $X\subset(\Gamma_f(\mathbb{D}))^*_{\mathbb{C}^2}$ and X is a “limit” (e.g. in the Hausdorff topology) of relatively closed analytic varieties in $V_p\times\mathbb{C}$?

Problem 4. Let f be analytic in $\mathbb{D}$ and suppose that $\Gamma_f(\mathbb{D})$ is not complete pluripolar. How big can the fiber of $(\Gamma_f(\mathbb{D}))^*_{\mathbb{C}^2}$ be over a “generic point” in $\pi_1((\Gamma_f(\mathbb{D}))^*_{\mathbb{C}^2})$? Can it be more than countable ? Does $(\Gamma_f(\mathbb{D}))^*_{\mathbb{C}^2}$ contain the graph of a reasonable function over a sufficiently massive subset of $\mathbb{C}$ which reflects certain generalized analytic continuation property of f ?

We have the following example in mind.

Example 5. By a segment we mean the (closed) part of a real line in $\mathbb{C}$ which joins two points in $\mathbb{C}$. Let U be as in Example 2. Consider a sequence of pairwise disjoint segments σ_n contained in U which accumulates to the whole circle $\partial\mathbb{D}$ and does not have other limit points in $\mathbb{C}$. Denote the endpoints of the segment σ_n by a_n and b_n and associate to σ_n the branch of the function $\sqrt{(z-a_n)(z-b_n)}$ on $\mathbb{C}\setminus\sigma_n$ which equals $z+O(1)$ near ∞. We will use this notation only for the mentioned branch. Let c_n be complex numbers so that $\sum|c_n|<\infty$. Let $\mathcal{F}(z)=\sum c_n\sqrt{(z-a_n)(z-b_n)}$ on $\mathbb{C}\setminus U$. The series converge uniformly on compacts in $\mathbb{C}\setminus U$. The function $f=\mathcal{F}|_{\mathbb{D}}$ has fine analytic continuation at each point of $\partial\mathbb{D}$. Moreover, $\mathcal{F}$ has the Mergelyan property on compacts in $\mathbb{C}\setminus U$. By Theorem 1 the graph $\Gamma_{\mathcal{F}}(\{|z|>1\}\setminus\overline{U})$ is in the pluripolar hull of $\Gamma_f(\mathbb{D})$. Since $\mathcal{F}$ has single-valued analytic continuation to $\{|z|>1\}\setminus\bigcup\sigma_n$, the graph of $\mathcal{F}$ over this set is contained in the pluripolar hull of $\Gamma_f(\mathbb{D})$ too.

Fix a number n. The graph of the function $\sqrt{(z-a_n)(z-b_n)}$ over $\mathbb{C}\setminus\sigma_n$ is an open subset of the algebraic curve $\{(z,w)\in\mathbb{C}^2: w^2=(z-a_n)(z-b_n)\}$. There is a neighborhood U_n of σ_n such that $\sum_{l\neq n}c_l\sqrt{(z-a_l)(z-b_l)}$ is analytic in U_n. Hence the analytic

set

$$A_n = \{(z,w) \in U_n \times \mathbb{C} : \big(w - \sum_{l \neq n} c_l \sqrt{(z-a_l)(z-b_l)}\big)^2 = c_n^2 (z-a_n)(z-b_n)\}$$

contains the graph $\Gamma_{\mathcal{F}}(U_n \setminus \sigma_n)$ and is therefore in the pluripolar hull of $\Gamma_{\mathcal{F}}(U_n \setminus \sigma_n)$ and, hence, of $\Gamma_f(\mathbb{D})$. The set A_n has two sheets over $U_n \setminus \sigma_n$, the second sheet is the graph of the function

$$-c_n \sqrt{(z-a_n)(z-b_n)} + \sum_{l \neq n} c_l \sqrt{(z-a_l)(z-b_l)}$$

This function has analytic continuation to $(\mathbb{C} \setminus \overline{\mathbb{D}}) \setminus \overline{U}$. Moreover, it has the Mergelyan property on compact subsets of $\mathbb{C} \setminus U$. By Theorem 2 it has fine analytic continuation at each point of $\partial\mathbb{D}$ to the disk $\mathbb{D}$. We obtained that the pluripolar hull of $\Gamma_f(\mathbb{D})$ contains a two-sheeted branched covering over the set $\mathbb{D} \cup (\mathbb{C} \setminus (\overline{\mathbb{D}} \cup \bigcup_{l \neq n} \sigma_l))$, the points a_n and b_n being the branch points. Repeating the argument for all other endpoints of the segments, we obtain that the pluripolar hull of $\Gamma_f(\mathbb{D})$ contains an infinitely-sheeted branched covering (countably many sheets over generic points) over the set $\mathbb{C} \setminus \partial\mathbb{D}$ with branch points $\{a_n\}$ and $\{b_n\}$. Note that the sheets over $\mathbb{D}$ are graphs of analytic functions. Moreover, the covering is unbranched over $\mathbb{C} \setminus (\mathbb{T} \cup \bigcup_n (\{a_n\} \cup \{b_n\}))$. The infinitely-sheeted branched covering over $\mathbb{C} \setminus \partial\mathbb{D}$ can be approximated by analytic subsets of $(\mathbb{C} \setminus \partial\mathbb{D}) \times \mathbb{C}$, the sheets of which over $\mathbb{C} \setminus \bigcup_1^n \sigma_l$ are the graphs of $\mathcal{F}_n(z) = \sum_{l=1}^n \pm c_l \sqrt{(z-a_l)(z-b_l)}$ with all possible choices of $+$ and $-$. Note that similar arguments as in Example 2 show that the function f is nowhere analytically extendible across $\partial\mathbb{D}$. Choosing the c_n more carefully we may reach that f is of class C^∞ on the closed unit disk.

When this paper was written we received a preprint of Zwonek [Zwo] where he constructed an analytic function in $\mathbb{D}$ which does not have analytic extension across $\partial\mathbb{D}$ and for which $(\Gamma_f(\mathbb{D}))^*_{\mathbb{C}^2}$ has at least two sheets over $\mathbb{D}$.

Problem 5. Is $(\Gamma_f(\mathbb{D}))^*_{\mathbb{C}^2}$ related to a suitable positive $(1,1)$-current ?

In Section 2 of the paper we will prove Proposition 1, Theorem 1 and its corollaries. In the remaining Section 3 we will prove Theorem 3.

2. Non-Trivial hull.

In this section we will prove Proposition 1, Theorem 1 and its corollaries. Recall that in Proposition 1 we consider domains D_i and D_e, $D_i \subset \mathbb{D}$ and $D_e \subset \mathbb{C} \setminus \overline{\mathbb{D}}$, such that the bounded components of $\mathbb{C} \setminus (\overline{D_i} \cup \overline{D_e})$ are similar rhombs $\Diamond_l$ for which the endpoints of one of the symmetry axis are the endpoints of a connected component of $\partial\mathbb{D} \setminus \mathcal{E}$. Here $\mathcal{E} \overset{def}{=} \overline{D_i} \cap \overline{D_e}$. The following Lemma will be useful.

Lemma 1. *Let $\bigcup_l I_l$ be a union of open pairwise disjoint arcs on $\partial\mathbb{D}$ which is thin at $p \in \partial\mathbb{D}$. Denote by $\bigcup_l \overline{\Diamond_l}$ the union of closed similar rhombs with the property that two opposite corners of $\overline{\Diamond_l}$ are the endpoints of I_l. Then $\bigcup \overline{\Diamond_l}$ is thin at p.*

For the proof we need the following proposition which is interesting in itself.

Proposition 2. *Let $E \subset \mathbb{C}$ be thin at the point $0 \in \overline{E}$. Let $T : \mathbb{C} \to \mathbb{C}$ be a mapping which satisfies a Lipschitz condition ($|Tz_1 - Tz_2| \leq C|z_1 - z_2|$ for $z_1, z_2 \in \mathbb{C}$) and such that $T(0) = 0$ and $|Tz| \geq c|z|$ for $z \in \mathbb{C}$. (C and c are positive constants). Then the set TE is thin at 0.*

This proposition was known already to Brelot. Since a slightly weaker assertion is stated in [Bre] (see chapter 7, paragraph 2) and we were not able to find an explicit reference for Proposition 2, we sketch the proof for the convenience of the reader.

Proof. Since E is thin at 0 there exists a subharmonic function $\mathcal{V}$ in a neighborhood of 0 with $\mathcal{V}(0) > -\infty$ and $\lim_{\xi\in E,\xi\to 0}\mathcal{V}(\xi) = -\infty$. Using the Riesz representation theorem and subtracting a harmonic function we may assume that $\mathcal{V}$ has the form

$$\mathcal{V}(z) = \int \log|\xi - z| d\mu(\xi)$$

for a positive Borel measure μ. Define the measure μ_1, $\mu_1(A) = \mu(T^{-1}(A))$ for each Borel set A, and put

$$\mathcal{V}_1(z) = \int \log|\xi - z| d\mu_1(\xi).$$

Then

$$\mathcal{V}_1(Tz) = \int \log|\xi - Tz| d\mu_1(\xi) = \int \log|T\xi - Tz| d\mu(\xi).$$

Hence $\mathcal{V}_1(Tz) \le \mathcal{V}(z) + \log C \cdot ||\mu||$ and $\mathcal{V}_1(0) \ge \mathcal{V}(0) + \log c \cdot ||\mu||$. □

Proof of Lemma 1. Note first that $\bigcup_l \overline{I_l}$ is thin at p if $\bigcup_l I_l$ is, since the sets differ by a countable (and hence thin) set. Denote by Λ the union of the boundaries of the rhombs. Let Λ_+ and Λ_- be the parts of Λ which are contained outside the unit disk and inside the closed unit disk, respectively. Both Λ_+ and Λ_-, can be represented as graphs over a part of $\partial\mathbb{D}$;

$$\Lambda_+ = \{r_+e^{i\phi} |\ r_+ = r_+(\phi),\ e^{i\phi} \in \bigcup_l \overline{I_l}\} \text{ and } \Lambda_- = \{r_-e^{i\phi} |\ r_- = r_-(\phi),\ e^{i\phi} \in \bigcup_l \overline{I_l}\}.$$

The mapping $T(e^{i\phi}) = r_+e^{i\phi}$ where $e^{i\phi} \in \bigcup_l \overline{I_l}$ can be extended to the whole plane as a Lipschitz continuous mapping which satisfies the conditions of Proposition 2 with 0 replaced by p. (The same is true for the mapping $T(e^{i\phi}) = r_-e^{i\phi}$). Since thinness is invariant under translation, Proposition 2 shows that both, Λ_+ and Λ_-, are thin at p. Therefore their union $\Lambda_+ \cup \Lambda_- = \Lambda$ is thin at p. Since the union of the boundaries of the closed rhombs is thin at p we conclude that the union of the closed rhombs is thin at p. □

We need the following immediate consequence of Lemma 1.

Corollary 4. *If $U = \bigcup_l \Diamond_l$ is thin at $p \in \partial\mathbb{D}$ then also $\overline{U} \setminus \overline{\mathbb{D}}$ is thin at p.*

Proof. Indeed, $\overline{U} \setminus \overline{\mathbb{D}} = \bigcup_l \overline{\Diamond_l} \setminus \overline{\mathbb{D}}$. □

We will make use also of the following three observations.

If the union of rhombs $\bigcup \overline{\Diamond_l}$ is thin at p there are arbitrarily small numbers $r > 0$ with the property that $\partial D(p, r) \cap \bigcup \overline{\Diamond_l} = \emptyset$. (See [Bre] or Proposition 2 with $Tz = |z|$ after suitable translation).

Moreover, looking at connected components of the union of closed intervals and replacing the previous rhombs by closed rhombs associated to these connected components in the same way as above, we may assume that the $\overline{\Diamond_l}$:s are pairwise disjoint.

If $\bigcup \overline{\Diamond_l}$ is thin at p then there exists another sequence of similar (open) rhombs $\Diamond'_j$ associated to disjoint open arcs I'_j of $\partial\mathbb{D}$, such that $\bigcup \Diamond'_j$ is thin at p and $\bigcup \overline{\Diamond_l} \subset \bigcup \Diamond'_j$. In fact, $\bigcup \overline{I_l}$ is thin at p. If for a subharmonic function $\mathcal{V}$ $\overline{\lim}_{z\in\bigcup \overline{I_l}, z\to p}\mathcal{V}(z) < a < \mathcal{V}(p)$, then for each $\overline{I_l}$ contained in a small neighborhood V_p of p $\sup_{\overline{I_l}} \mathcal{V} < a$, hence $\sup_{\widetilde{I_l}} \mathcal{V} < a$ for some open arc $\widetilde{I_l} \supset \overline{I_l}$. Take also for the other arcs $\overline{I_l}$ suitable open arcs $\widetilde{I_l}$ containing them. The set $\bigcup_l \widetilde{I_l}$ is thin at p. Let the I'_j be the connected components of the latter union and associate rhombs $\Diamond'_j$ to them.

Proof of Proposition 1. We will assume that the $\Diamond_l$ in the statement of Proposition 1 are pairwise disjoint (shrinking otherwise the sets D_i and D_e) and prove fine analytic continuation to $\overline{D(p, r)} \setminus \bigcup_j \Diamond'_j$ for a suitable small $r > 0$ and the rhombs $\Diamond'_j$ described

above. We may assume that $r>0$ is chosen so that $\partial D(p,r)$ does not meet $\bigcup_j \Diamond'_j$ (by the remark above). Keep notation $\overline{\Diamond_l}$ for only those of the original rhombs which are contained in $D(p,r)$ and $\Diamond'_j$ for those of the enlarged rhombs which are contained there. The function f is analytic in each of the domains $\mathcal{D}'_i \overset{def}{=} D(p,r)\cap\mathbb{D}\setminus\bigcup_l\overline{\Diamond_l}$ and $\mathcal{D}'_e \overset{def}{=} D(p,r)\cap(\mathbb{C}\setminus\overline{\mathbb{D}})\setminus\bigcup_l\overline{\Diamond_l}$ and Hölder continuous in the union of the closures. Both domains have rectifiable boundary, hence by Cauchy's formula

$$f(z)=\frac{1}{2\pi i}\int_{\partial\mathcal{D}'_i}\frac{f(\xi)}{\xi-z}d\xi+\frac{1}{2\pi i}\int_{\partial\mathcal{D}'_e}\frac{f(\xi)}{\xi-z}d\xi,\qquad z\in\mathcal{D}'_i\cup\mathcal{D}'_e.$$

The contours of integration are always oriented as boundaries of relatively compact domains. Note that one of the integrals in the sum above will be equal to zero. Using that $\partial\mathbb{D}\cap D(p,r)\setminus\bigcup\Diamond_l=\mathcal{E}\cap D(p,r)=\partial\mathcal{D}'_i\cap\partial\mathcal{D}'_e$ and orientation over this set is provided twice with opposite orientation we obtain

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_{\partial D(p,r)}\frac{f(\xi)}{\xi-z}d\xi-\frac{1}{2\pi i}\sum_l\int_{\partial\Diamond_l}\frac{f(\xi)}{\xi-z}d\xi \overset{def}{=}\\ &\overset{def}{=} J(z)-\sum_l J_l(z),\qquad z\in\mathcal{D}'_i\cup\mathcal{D}'_e.\end{aligned}$$

By Privalov's theorem $J(z)$ extends to a Hölder continuous function of order α in $\overline{D(p,r)}$ if $\alpha<1$ and of any order less than 1 if $\alpha=1$. The measure $f(\xi)d\xi$ on $\bigcup\partial\Diamond_l$ is a finite Borel measure concentrated on a subset of $\bigcup\Diamond'_j$. To prove Proposition 1 let $\kappa_n=\bigcup_{l=1}^n\overline{\Diamond_l}$ and $\mathcal{F}_n(z)=J((1-1/n)z)-\sum_{l=1}^n J_l(z)$, $z\in\overline{D(p,r)}\setminus\kappa_n$. We have to check that the $\mathcal{F}_n$ converge uniformly to f on $\overline{\mathcal{D}}'_i\cup\overline{\mathcal{D}}'_e=\overline{D(p,r)}\setminus\bigcup_j\Diamond'_j$. To obtain a uniform estimate of J_l on $\overline{D(p,r)}\setminus\Diamond_l$ we use that for $z\notin\overline{\Diamond_l}$ the Cauchy type integral of the constant function $f(z)$ with pole at z along $\partial\Diamond_l$ vanishes. We get for $z\in\overline{D(p,r)}\setminus\overline{\Diamond_l}$

$$|J_l(z)|=|\frac{1}{2\pi i}\int_{\partial\Diamond_l}\frac{f(\xi)-f(z)}{\xi-z}d\xi|\le C\int_{\partial\Diamond_l}\frac{|\xi-z|^\alpha}{|\xi-z|}|d\xi|.$$

Let $N>n$ be a natural number and let $z\in\overline{D(p,r)}\setminus\bigcup_{l=n}^N\overline{\Diamond_l}$. Then

$$\sum_{n\le l\le N}|J_l(z)|\le C\int_{\bigcup_{n\le l\le N}\partial\Diamond_l}\frac{|\xi-z|^\alpha}{|\xi-z|}|d\xi|\le C\int_{\bigcup_{l\ge n}\partial\Diamond_l}\frac{|\xi-z|^\alpha}{|\xi-z|}|d\xi|.$$

Represent the contour of integration on the right hand side as the union of its part Λ_k^- contained in $\overline{\mathbb{D}}$ and its part Λ_k^+ contained in $\mathbb{C}\setminus\overline{\mathbb{D}}$. Each of the parts is the graph of a Lipschitz continuous function over a subset of $(\partial\mathbb{D})\cap\overline{D(p,r)}$ with uniform estimate for the Lipschitz constant (which depends on the angle of the truncated non-tangential cones contributing to D_i and D_e). For a point $\zeta\ne 0$ we denote by ζ' its radial projection to the circle, $\zeta'=\zeta/|\zeta|$. Using the inequality $|\xi-z|\ge \mathrm{const}|\xi'-z'|$ and estimating the arc-length on Λ_k^+ and on Λ_k^- by arc-length of the radial projection we obtain

$$\begin{aligned}\sum_{n\le l\le N}|J_l(z)| &\le C\int_{\partial\mathbb{D}\cap\bigcup_{l\ge n}\overline{\Diamond_l}}|\xi'-z'|^{\alpha-1}|d\xi'|\le\\ &\le C'\int_{\gamma_n}|e^{i\phi}-1|^{\alpha-1}|de^{i\phi}|,\qquad z\in\overline{D(p,r)}\setminus\bigcup_{l=n}^N\overline{\Diamond_l}.\end{aligned}$$

where γ_n is the arc of the circle which is symmetric around the point 1 and has length $\mathrm{mes}_1(\partial\mathbb{D}\cap\bigcup_{l\ge n}\overline{\Diamond_l})$. Since $\alpha>0$ the right hand side converge to zero for $n\to\infty$. This proves the proposition. □

For a domain $G\subset\mathbb{C}$, a Borel subset $\mathcal{E}$ of ∂G and a point $z\in G$ we denote by $\omega(z,\mathcal{E},G)$ the harmonic measure of $\mathcal{E}$ with respect to G computed at the point z.

Proof of Theorem 1. Let $V = \overline{D(p,r)} \setminus U$, where $U \subset \mathbb{C} \setminus \overline{\mathbb{D}}$ is open and thin at p. We will first obtain a harmonic measure estimate. Let $\rho > 0$ be small enough and such that $\{|z-p| = \rho\} \cap U = \emptyset$. Since U is thin at p such ρ exists. Let J be a closed subarc of $\partial D(p,\rho)$ contained in $\mathbb{D}$. Decreasing ρ we may assume that the length of J is at least $5\pi\rho/6$. Let K_n be an increasing sequence of compact subsets of U, each K_n being the finite disjoint union of closures of simply connected domains. Then $D(p,r) \setminus K_n$ is connected. We claim that if ρ is small enough there exists a number r_1, $0 < r_1 < \rho$ and an open set $U_1 \supset U \cap D(p,\rho)$, which is thin at p such that the following harmonic measure estimate holds:

$$\omega(z, J, D(p,\rho) \setminus K_n) \geq \frac{1}{4} \text{ for each } z \in V_1 = D(p,r_1) \setminus U_1 \text{ and each } n. \tag{2.1}$$

In fact, since U is thin at p there is a subharmonic function $\mathcal{U}$ in a neighborhood of p which is finite at p and tends to $-\infty$ along the set U. Taking ρ small enough and adding a constant to $\mathcal{U}$ we may assume that $\mathcal{U}$ is defined and < 0 in $\overline{D(p,\rho)}$. Multiplying $\mathcal{U}$ by a positive constant we may assume that $\mathcal{U}(p) > -1/12$. Taking ρ small enough, we may assume that $\mathcal{U} < -1$ on $U \cap \overline{D(p,\rho)}$. Then

$$\omega(z, J, D(p,\rho) \setminus K_n) \geq \omega(z, J, D(p,\rho)) + \mathcal{U}(z), \qquad z \in D(p,\rho) \setminus K_n. \tag{2.2}$$

Indeed, the boundary of $D(p,\rho) \setminus K_n$ is smooth, hence regular for the Dirichlet problem. The left hand side is harmonic in this domain and extends continuously to all but two points of its closure, the right hand side is subharmonic in the domain, bounded from above and its boundary values at all but two points are majorized by those on the left hand side. Denote by U_1 the set $U_1 \stackrel{def}{=} \{z \in D(p,\rho) : \mathcal{U}(z) < -1/12\}$. U_1 is open and since $\mathcal{U}(p) > -1/12$ the set U_1 is thin at p. Clearly $U_1 \supset U \cap D(p,\rho)$. By the assumption on the length of J we have $\omega(p, J, D(p,\rho)) \geq 5/12$. Let $r_1 \in (0,\rho)$ be so small so that

$$\omega(z, J, D(p,\rho)) > \frac{4}{12} \text{ for } z \in D(p,r_1). \tag{2.3}$$

Then by (2.2), the definition of U_1 and by (2.3)

$$\omega(z, J, D(p,\rho) \setminus K_n) > \frac{4}{12} - \frac{1}{12} = \frac{1}{4} \text{ for } z \in D(p,r_1) \setminus U_1 \stackrel{def}{=} V_1 \text{ for each } n.$$

(2.1) is proved.

Suppose now that f has fine analytic continuation $\mathcal{F}$ at p to a fine neighborhood $V = \overline{D(p,r)} \setminus U$, i.e. there exist analytic functions $\mathcal{F}_n$ in neighborhoods $U(\mathcal{F}_n)$ of V which converge uniformly to $\mathcal{F}$ on V. Shrinking the neighborhoods $U(\mathcal{F}_n)$ we may always assume that $\sup_{U(\mathcal{F}_n)} |\mathcal{F}_n| \leq C$ for all n and a constant $C > 1$. Since U is simply connected one can choose an increasing sequence of compact subsets K_n of U, each being the finite disjoint union of closures of simply connected domains such that $\overline{D(p,r) \setminus K_n} \subset U(\mathcal{F}_n)$. Hence $\mathcal{F}_n$ are analytic in $D(p,r) \setminus K_n$ and continuous in $\overline{D(p,r) \setminus K_n}$ and their maximum norms in these sets are bounded by the constant C. Take ρ, r_1 and V_1 as above. Fix an arbitrary point $z \in V_1$ and define $Q_{n,z}(\xi) = \mathcal{F}_n(\xi) + \mathcal{F}(z) - \mathcal{F}_n(z)$, $\xi \in \overline{D(p,r) \setminus K_n}$. Then $Q_{n,z}$ are analytic and uniformly bounded on $D(p,\rho) \setminus K_n$ and continuous on $\overline{D(p,\rho) \setminus K_n}$, $Q_{n,z}(z) = \mathcal{F}(z)$, and $Q_{n,z} \to \mathcal{F}$ uniformly on V for $n \to \infty$.

Let B be a ball in $\mathbb{C}^2$ which contains the graphs $\Gamma_{Q_{n,z}}(D(p,r) \setminus K_n)$ for all n and z. Fix $z \in V_1$. Let u be a plurisubharmonic function in $\mathbb{C}^2$ which equals $-\infty$ on $\Gamma_f(\mathbb{D})$. Adding a constant, we may assume that $u < 0$ in B. $J \subset V \cap \mathbb{D}$ and for large n the set $\Gamma_{Q_{n,z}}(J)$ is uniformly close to $\Gamma_f(J) = \Gamma_{\mathcal{F}}(J)$. Since u is upper semi-continuous for each large N there exists n such that $u(\xi, Q_{n,z}(\xi)) < -N$ for $\xi \in J$. The function $\xi \mapsto u(\xi, Q_{n,z}(\xi))$ is a negative subharmonic function on $D(p,\rho) \setminus K_n$ which is upper semi-continuous on the closure of this set, hence by the estimate of harmonic measure we obtain

$$u(z, \mathcal{F}(z)) = u(z, Q_{n,z}(z)) \leq -N\omega(z, J, D(p,\rho) \setminus K_n) < -\frac{N}{4}. \tag{2.4}$$

Since N was arbitrary we obtain $u(z, \mathcal{F}(z)) = -\infty$ for all $z \in V_1$ if $u = -\infty$ on $\Gamma_f(\mathbb{D})$.

Suppose now that $\text{Int}V \setminus \overline{\mathbb{D}}$ has a connected component $\overset{\circ}{V}$ which is non-thin at p. Then $\overset{\circ}{V} \cap V_1 = \overset{\circ}{V} \cap \overline{D(p, r_1)} \setminus U_1$ is non-thin at p (since $\overset{\circ}{V} \subset (\overset{\circ}{V} \cap \overline{D(p, r_1)} \setminus U_1) \cup \{|z-p| > r_1\} \cup U_1$ and the last two sets are thin at p). Hence $\overset{\circ}{V} \cap V_1$ is not polar, and therefore, since $\mathcal{F}$ is analytic on $\overset{\circ}{V}$, $\Gamma_{\mathcal{F}}(\overset{\circ}{V})$ is contained in the pluripolar hull of $\Gamma_f(\mathbb{D})$.

Suppose $\overline{U} \setminus \overline{\mathbb{D}}$ is thin at p. There exist arbitrarily small numbers $\rho > 0$ such that $\{|z-p| = \rho\}$ does not meet $\overline{U} \setminus \overline{\mathbb{D}}$, hence for those ρ the set $\{|z-p| = \rho\} \cap \{|z| > 1\}$ is contained in the complement of $\overline{U} \setminus \overline{\mathbb{D}}$ in $\mathbb{C} \setminus \overline{\mathbb{D}}$, namely in $\text{Int}V \cap \{|z| > 1\}$. There cannot be two disjoint open connected subsets of $\mathbb{C} \setminus \overline{\mathbb{D}}$ for which p is an accumulation point, which both contain half-circles $\{|z-p| = \rho\} \cap \{|z| > 1\}$ for some positive numbers ρ. Since the open set $\mathbb{C} \setminus (\overline{\mathbb{D}} \cup \overline{U})$ is not thin at p it has exactly one such component and this component is non-thin at p. Theorem 1 is proved. □

The proof of Theorem 2 is a slight modification of the proof of Theorem 1. We will omit it.

Proposition 3. *Suppose the continuous function $\mathcal{F}$ on a closed neighborhood $V = \overline{D(p,r)} \setminus U$ of p is finely analytic at p. Suppose $\gamma : [-1, 1] \to \mathbb{C}$ is a smooth arc with $\gamma(0) = p$ which divides $\overline{D(p,r)}$ into two connected components $D_+(p,r)$ and $D_-(p,r)$. Then there is a smaller fine neighborhood V_1 of p such that $\mathcal{F}|_{D_+(p,r) \cap V_1}$ is uniquely determined by $\mathcal{F}|_{D_-(p,r) \cap V}$.*

Proof. It is enough to show that if $\mathcal{F}$ is finely analytic at p on V and $\mathcal{F}|_{D_-(p,r) \cap V} \equiv 0$ then $\mathcal{F}|_{V_1}$ is equal to zero for some fine neighborhood $V_1 \subset V$ of p. As in the proof of Theorem 1 there exist compact subsets K_n of U, each being the finite disjoint union of closures of simply connected domains and analytic functions $\mathcal{F}_n$ on $D(p, r) \setminus K_n$ which are continuous on $\overline{D(p, r) \setminus K_n}$ and uniformly bounded by a constant $C > 1$, and converge to $\mathcal{F}$ uniformly on V. Let J be a closed arc of a circle $\partial D(p, \rho)$ for some $\rho > 0$ which is contained in $D_-(p, r) \cap V$ and has length at least $5\pi\rho/6$. Since $\mathcal{F} = 0$ on J, the numbers $\epsilon_n \overset{def}{=} \max_J |\mathcal{F}_n|$ are less than 1 for $n > n_0$ and tend to zero for $n \to \infty$. The same arguments as in the proof of Theorem 1 give a number $r_1 > 0$ and an open set U_1 which is thin at p and a harmonic measure estimate analogously to (2.1) such that the Two-constant Theorem gives for $z \in V_1 = \overline{D(p, r_1)} \setminus U_1$ and all $n > n_0$

$$\begin{aligned}
\log |\mathcal{F}_n(z)| &\leq \log \epsilon_n \cdot \omega(z, J, D(p, \rho) \setminus K_n) + \\
&+ \log C \cdot (1 - \omega(z, J, D(p, \rho) \setminus K_n)) \leq \\
&\leq \log \epsilon_n \cdot \frac{1}{4} + \log C.
\end{aligned}$$

Hence $\mathcal{F}(z) = \lim_{n\to\infty} \mathcal{F}_n(z) = 0$ for $z \in V_1$. □

Proof of Corollary 1. By Proposition 1, $f|_{D_i}$ has fine analytic continuation to a set $V_1 = \overline{D(p, r_1)} \setminus U_1 \subset \overline{D}_i \cup \overline{D}_e$ for some $r_1 > 0$ where U_1 is an open set which is thin at p. Moreover, this fine analytic continuation equals $f|_{V_1}$. The domain D_e is non-thin at p, since D_e contains $D(p, \rho) \cap \{|z| > 1\} \setminus \overline{U}$, where $\overline{U}$ is thin at p. By Theorem 1 $\Gamma_f(V_1)$ is in the pluripolar hull of $\Gamma_f(D_i)$, and hence, as in the proof of the second part of Theorem 1, $\Gamma_f(D_e)$ is contained in the pluripolar hull of $\Gamma_f(D_i)$. □

Proof of Corollary 2: Let $\bigcup_l I_l$ be a union of disjoint open arcs on $\partial\mathbb{D}$ such that $\bigcup_l I_l$ is dense on $\partial\mathbb{D}$, its linear measure is $< 2\pi$ and $\bigcup_l I_l$ is thin at 1. Let $G \subset \mathbb{D}$ be a Jordan domain whose boundary γ is a smooth, nowhere analytic curve with $\partial\mathbb{D} \setminus \gamma \subset \bigcup_l \Diamond_l$, where the $\Diamond_l$:s are similar rhombs corresponding to the arcs I_l like in Section 1. Let f be a conformal mapping of G onto $\mathbb{D}$. f extends to a smooth homeomorphism of $\overline{G}$ onto $\overline{\mathbb{D}}$. Since γ is nowhere analytic f and its inverse f^{-1} do not have analytic continuation across

any part of the boundary of their domain of definition. However by Schwarz reflection principle f admits pseudocontinuation across the set $\mathcal{E} = \partial\mathbb{D} \setminus \bigcup_l I_l$ and hence extends to a function in $A(\overline{D_i} \cup \overline{D_e})$ for suitable domains D_i and D_e of the kind described before the statement of Theorem 1. Note that the extended function is also univalent. By Corollary 1 the pluripolar hull of $\Gamma_f(D_i)$ (hence of $\Gamma_f(G)$) contains $\Gamma_f(D_e)$. The graph of f over the subset G in the z-plane, $\Gamma_f(G) = \{(z,w) \in \mathbb{C}^2 | z \in G, w = f(z)\}$, can be considered as the graph of its inverse function over the set $\mathbb{D}$ in the w-plane, $\{(z,w) \in \mathbb{C}^2 | w \in \mathbb{D}, z = f^{-1}(w)\}$. The Corollary follows. $\square$

3. Points which are not in the pluripolar hull

In this Section we will prove Theorem 3. For the proof it will be convenient to use the following known results.

Let Ω be a pseudoconvex domain in $\mathbb{C}^N$. In [LePo] the *negative pluripolar hull* is defined as

$$E^-_\Omega \stackrel{def}{=} \bigcap \{z \in \Omega : u(z) = -\infty\},$$

where the intersection is taken over all *negative* plurisubharmonic functions in Ω that are $-\infty$ on E. The following relation between the negative pluripolar hull and the pluripolar hull holds (see [LePo]).

Theorem 5. *Let Ω be a pseudoconvex domain in $\mathbb{C}^N$. Let $\{\Omega_j\}$ be an increasing sequence of relatively compact subdomains of Ω with $\bigcup_j \Omega_j = \Omega$. Let $E \subset \Omega$ be pluripolar. Then*

$$E^*_\Omega = \bigcup_j (E \cap \Omega_j)^-_{\Omega_j}.$$

For a subset $E \subset \Omega$, the *pluriharmonic measure* at a point $z \in \Omega$ of E relative to Ω, is defined as

$$W(z, E, \Omega) = -\sup\{u(z) : u \text{ is plurisubharmonic in } \Omega \text{ and u} \leq -\chi_E\}, \tag{3.1}$$

where χ_E is the characteristic function of the set E. The relation between the negative pluripolar hull and pluriharmonic measure is given in the following Theorem [LePo].

Theorem 6. *Let Ω be a domain in $\mathbb{C}^N$ and let $E \subset \Omega$ be pluripolar. Then*

$$E^-_\Omega = \{z \in \Omega : W(z, E, \Omega) > 0\}.$$

The following theorem was recently proved by Wiegerinck and Edigarian in [EdWi3].

Theorem 7. *Let Ω be a pseudoconvex open set in $\mathbb{C}^N$ and let $E \subset \Omega$ be a F_σ pluripolar subset. Assume that E is connected. Then E^*_Ω is also connected.*

In [Zer] A. Zeriahi proved the following Theorem:

Theorem 8 (Zeriahi)**.** *Let Ω be a pseudoconvex domain and F a pluripolar subset of Ω of type F_σ and let E be a closed subset of Ω such that $E \supset F^*_\Omega$. Then there exists a plurisubharmonic function u on Ω which is continuous and $> -\infty$ on $\Omega \setminus E$ and $= -\infty$ on F.*

For a natural number j we denote by $\mathbb{B}_j$ the (open) ball of radius j and center 0 in $\mathbb{C}^2$. The key in the proof of Theorem 3 is contained in the following Lemma.

Lemma 2. *Let f be analytic in $\mathbb{D}$ and let K be a closed disk in $\mathbb{D}$. Choose j_0 so that $\Gamma_f(K) \subset \mathbb{B}_{j_0}$. Put $E = \overline{(\Gamma_f(\mathbb{D}))^*_{\mathbb{C}^2}}$, let $j \geq j_0$ and consider the compact set $E_j = E \cap \overline{\mathbb{B}_j}$ and the open set $U_j = \mathbb{C} \setminus \pi_1(E_j)$. Suppose $p \in \partial\mathbb{D}$ is non-thin for U_j. Then for each $\epsilon > 0$ there exists a continuous plurisubharmonic function g in $\mathbb{B}_j$, such that $g \leq 0$, $g = -1$ on $\Gamma_f(K)$ and $g(p,w) \geq -\epsilon$ for any $w \in \mathbb{C}$ such that $(p,w) \in \mathbb{B}_j$. In particular*

$$W((p,w), \Gamma_f(K), \mathbb{B}_j) = 0$$

for each $w \in \mathbb{C}$ such that $(p,w) \in \mathbb{B}_j$.

Corollary 5. *Let f be analytic in $\mathbb{D}$ and let $p \in \partial\mathbb{D}$. Then either $\{p\} \times \mathbb{C}$ does not meet $(\Gamma_f(\mathbb{D}))^*_{\mathbb{C}^2}$ or $\pi_1(\,\overline{(\Gamma_f(\mathbb{D}))^*_{\mathbb{C}^2}}\,)$ is a fine neighborhood of p.*

The Corollary improves a result of [EdWi3].

Proof of Corollary 5. Suppose $\pi_1(E) = \pi_1(\,\overline{(\Gamma_f(\mathbb{D}))^*_{\mathbb{C}^2}}\,)$ is not a fine neighborhood of p, i.e. $U = \mathbb{C} \setminus \pi_1(E)$ is not thin at p. Let K be a closed disk contained in $\mathbb{D}$ and let j_0 be so large that $\Gamma_f(K) \subset \mathbb{B}_{j_0}$. Let $j \geq j_0$. Since $E_j = E \cap \overline{\mathbb{B}_j} \subset E$ the set $\pi_1(E_j)$ is also not a fine neighborhood of p, i.e. the set $U_j = \mathbb{C} \setminus \pi_1(E_j)$ is not thin at p. By Lemma 2 and Theorem 6 for each $j \geq j_0$ the set $(\{p\} \times \mathbb{C}) \cap \mathbb{B}_j$ does not meet the set $(\Gamma_f(K))^-_{\mathbb{B}_j}$. By Theorem 5 $\{p\} \times \mathbb{C}$ does not meet $(\Gamma_f(K))^*_{\mathbb{C}^2}$. Since the $\mathbb{C}^2$-pluripolar hull of $\Gamma_f(K)$ and $\Gamma_f(\mathbb{D})$ coincide the Corollary is proved. □

Note that the arguments of Lemma 2 may be applied in situations when $\mathbb{D}$ is replaced by another planar domain. In particular if $\pi_1(E)$ is the closure of a domain, this gives a tool to study $(\Gamma_f(\mathbb{D}))^*_{\mathbb{C}^2}$ over boundary points of the domain. We will not work this out here.

Proof of Theorem 3. Suppose, on the contrary, that for each point $p \in \partial\mathbb{D}$ the set $U = \mathbb{C} \setminus \pi_1(E)$ is non-thin at p. By Corollary 5 the set $\{p\} \times \mathbb{C}$ does not meet $(\Gamma_f(\mathbb{D}))^*_{\mathbb{C}^2}$ for each $p \in \partial\mathbb{D}$. Hence by Theorem 7 $(\Gamma_f(\mathbb{D}))^*_{\mathbb{C}^2} \subset \mathbb{D} \times \mathbb{C}$. □

Proof of Lemma 2. In case $p \in U_j$ we decrease U_j by replacing it by a simply connected set $\widetilde{U}_j$ with $p \in \partial\widetilde{U}_j$ which is still not thin at p (e.g. cut the connected component of U_j containing p along a curve joining p with the boundary ∂U_j). Put $\widetilde{U}_j = U_j$ if $p \notin U_j$. Then $\widetilde{U}_j \subset U_j$ is not thin at p, $p \notin \widetilde{U}_j$ and $\widetilde{U}_j$ is simply connected. Let $K_n \subset \widetilde{U}_j$ be an increasing sequence of compacts, each of them being the finite disjoint union of closures of smoothly bounded simply connected domains, such that $\widetilde{U}_j = \bigcup K_n$. Then for each natural number n the set $D_n = \widehat{\mathbb{C}} \setminus (K \cup K_n)$ is a domain. Note that D_n has regular boundary for the Dirichlet problem. Hence the harmonic measure $\omega(z, \partial K, D_n)$ of ∂K for the domain D_n computed at the point $z \in \overline{D}_n$ is a continuous function on $\overline{D}_n$ which is harmonic on D_n.

We claim that

$$\lim_{n\to\infty} \omega(p, \partial K, D_n) = 0. \tag{3.2}$$

Indeed, extend for each n the function $\omega(z, \partial K, D_n)$ to the set K_n by putting it equal to zero there. Denote the extended function by h_n. h_n is continuous and subharmonic on $\widehat{\mathbb{C}} \setminus K$, it is non-negative and $h_{n+1} \leq h_n$ for each n. Hence $h = \lim_{n\to\infty} h_n$ is non-negative and subharmonic on $\widehat{\mathbb{C}} \setminus K$ (being the decreasing limit of a sequence of subharmonic functions). Moreover, $h = 0$ on $\widetilde{U}_j$. Since $\widetilde{U}_j$ is non-thin at p, p is an accumulation point of $\widetilde{U}_j$ and $h(p) = 0$. Hence the claim.

Let now $\epsilon > 0$ and choose n so that

$$\omega(z, \partial K, D_n) < \epsilon. \tag{3.3}$$

The function

$$v(z, w) = \begin{cases} -\omega(z, \partial K, D_n), & z \in D_n, (z, w) \in \mathbb{B}_j \\ -1, & z \in K, (z, w) \in \mathbb{B}_j \end{cases} \tag{3.4}$$

is plurisubharmonic on a large part of the ball $\mathbb{B}_j$, precisely on $\{(z, w) \in \mathbb{B}_j : z \notin K_n\}$. We want to obtain a plurisubharmonic function in the whole $\mathbb{B}_j$ using a standard gluing procedure near $(\partial K_n \times \mathbb{C}) \cap \mathbb{B}_j$ together with the Theorem of Zeriahi. Denote by u a plurisubharmonic function on $\mathbb{B}_{j+1}$ which is continuous and $> -\infty$ on $\mathbb{B}_{j+1} \setminus E$ (with $E = \overline{(\Gamma_f(\mathbb{D}))^*_{\mathbb{C}^2}}\,$) and equals $-\infty$ on $\Gamma_f(\mathbb{D})$. Its existence is guaranteed by Zeriahi's Theorem. Adding a constant we may assume that $u < 0$ on the compact set $\overline{\mathbb{B}_j} \subset \mathbb{B}_{j+1}$.

Let $G_n \subset \widetilde{U}_j$ be a smoothly bounded open set such that $K_n \subset G_n \subset \overline{G_n} \subset \widetilde{U}_j$. (We think of G_n being close to K_n.) Let V_n be a small neighborhood of ∂G_n such that $\partial G_n \subset V_n \subset \overline{V_n} \subset \widetilde{U}_j \setminus K_n \subset D_n$. Then there exists a positive constant δ such that $-\omega(z, \partial K, D_n) < -\delta$ on $\overline{V_n}$. Since $\overline{V_n} \subset U_j = \mathbb{C} \setminus \pi_1(E_j)$, the set $(\overline{V_n} \times \mathbb{C}) \cap \overline{\mathbb{B}}_j$ does not meet E, hence the function u is $> -\infty$ and continuous on this set. Multiplying u by a positive constant we may assume that $u > -\delta$ on the compact subset $(\overline{V_n} \times \mathbb{C}) \cap \overline{\mathbb{B}_j}$. For $(z, w) \in \mathbb{B}_j$ define

$$g(z,w) = \begin{cases} u(z,w), & z \in G_n, (z,w) \in \mathbb{B}_j \\ \max\{v(z,w), u(z,w)\}, & z \notin G_n, (z,w) \in \mathbb{B}_j. \end{cases} \tag{3.5}$$

v is defined on $\{(z,w) \in \mathbb{B}_j : z \notin K_n\}$ and $K_n \subset G_n$, hence g is well defined. On $(\overline{V_n} \times \mathbb{C}) \cap \mathbb{B}_j$ we have the inequality $v < -\delta < u$, hence $g = u$ on $(V_n \times \mathbb{C}) \cap \mathbb{B}_j$. Since u and v are plurisubharmonic where they are defined, the function g is plurisubharmonic on $\mathbb{B}_j$. Since for $(z, w) \in \Gamma_f(K) \cap \mathbb{B}_j$ the relations $u(z, w) = -\infty$, $v(z, w) = -1$ hold we obtain for these points $g(z, w) = -1$. On the other hand, since $p \notin G_n$, for points of the form $(p, w) \in \mathbb{B}_j$ we have by (3.3)

$$g(p,w) \geq v(p,w) > -\epsilon$$

The Lemma is proved. □

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